

GLOBAL-IN-TIME STRICHARTZ ESTIMATES AND CUBIC SCHRÖDINGER EQUATION ON METRIC CONE

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ABSTRACT. We study the Strichartz estimates for Schrödinger operator \mathcal{L}_V on metric cone X , where the metric cone $X = C(Y) = (0, \infty)_r \times Y$ and the cross section Y is a compact $(n-1)$ -dimensional Riemannian manifold (Y, h) . The equipped metric on X is given by $g = dr^2 + r^2h$, and let Δ_g be the Friedrich extension positive Laplacian on X and $V = V_0r^{-2}$ where $V_0 \in C^\infty(Y)$ such that $\Delta_h + V_0 + (n-2)^2/4$ is a strictly positive operator on $L^2(Y)$. We establish the full range of the global-in-time Strichartz estimate without loss for the Schrödinger equation associated with the operator $\mathcal{L}_V = \Delta_g + V_0r^{-2}$ except for the endpoint inhomogeneous inequality. As an application, we study the well-posed theory and scattering theory for the cubic nonlinearity Schrödinger equation on this setting.

Key Words: Local smoothing estimate, metric cone, Strichartz estimate, scattering theory

AMS Classification: 58J47, 42B37, 35Q40, 47J35.

1. INTRODUCTION AND STATEMENT OF MAIN RESULT

We first study the Strichartz estimates for the solution of Schrödinger equations on the setting of metric cone and then use it to study the well-posed theory and scattering theory for a nonlinear Schrödinger equation. Let (Y, h) be a compact $(n-1)$ -dimensional Riemannian manifold, the metric cone $X = C(Y)$ considered here is of the form $(0, \infty)_r \times Y$, and the metric of X is $g = dr^2 + r^2h$. Let Δ_g denote the Friedrichs extension of Laplace-Beltrami from the domain $C_c^\infty(X^\circ)$, compactly supported smooth functions on the interior of the metric cone. Consider the Schrödinger operator $\mathcal{L}_V = \Delta_g + V$ where $V = V_0(y)r^{-2}$ and $V_0(y)$ is a smooth function on the section Y . A simplest example of a metric cone is the Euclidean space \mathbb{R}^n whose cross section is \mathbb{S}^{n-1} with the round metric. However we remark that the general metric cones do not have many symmetries, e.g. rotation, analogous to that of Euclidean space but still have dilation symmetry. Hence the Schrödinger operator considered here \mathcal{L}_V is homogeneous of degree -2 .

Metric cones were first studied from the perspective of wave diffraction from the cone point; see [10, 11, 31]. Cheeger and Taylor [5, 6] studied the Laplacian defined on cones from the functional calculus. Other aspects on Schrödinger operator on the metric cone, even with inverse-square potentials that are homogeneous of degree -2 , also have been extensively studied; for example, the asymptotical behavior of Schrödinger propagator was considered in [35]; the heat kernel and Riesz transform kernel were studied in [16, 24]; regularity properties of wave propagation [25]; the restriction estimate for Schrödinger solution by the first author [37].

In this paper, we consider a spacetime-type estimate of the solution $u : \mathbb{R} \times X \rightarrow \mathbb{C}$ to the initial value problem (IVP) for the Schrödinger equation on metric cone X ,

$$(1.1) \quad i\partial_t u(t, z) + \mathcal{L}_V u(t, z) = 0, \quad u(t, z)|_{t=0} = u_0(z), \quad (t, z) \in \mathbb{R} \times X.$$

We are particularly interested in the Strichartz estimates for the solution of Schrödinger equations on the metric cone since the Strichartz estimates are a powerful tool for studying the behaviour of solutions to nonlinear dispersive equations, e.g. Schrödinger equation. More precisely, let u be the solution to (1.1), we aim to establish the inequality in the form of

$$\|u(t, z)\|_{L_t^q L_z^r(I \times X)} \leq C \|u_0\|_{H^s(X)},$$

where I is a subset interval of \mathbb{R} and H^s denotes the L^2 -Sobolev space over X , and (q, r) is an *admissible pair*, i.e.

$$(1.2) \quad (q, r) \in \Lambda_0 := \{2 \leq q, r \leq \infty, \quad 2/q + n/r = n/2, \quad (q, r, n) \neq (2, \infty, 2)\}.$$

There is a large number of work studying Strichartz inequalities on Euclidean space or manifolds, we can not cite all of the papers here but refer the reader to see [15, 20, 28, 29] and references therein. In particular, we mention a few of the most relevant ones about the Strichartz estimate for the Schrödinger equation on exact cone or with perturbation of inverse-square potentials or the slightly different setting of asymptotically conic manifold. We recall that an asymptotically conic manifold M , outside some compact set, is isometric to the conical space X far away the cone tip. Thus the problems on each manifold are closely related. On the non-trapping asymptotically conic manifold M , the local-in-time Strichartz estimates were established in Hassell, Tao and Wunsch [17, 18] and Mizutani [23]. Hassell and the first author [19] improved to the global-in-time Strichartz inequality and fixed the endpoint one by exploring the decay and oscillation behavior of the micro-localized spectral measure. Burq, Guillarmou and Hassell [2] proved a local-in-time Strichartz estimate without loss on the asymptotically conic manifold with a trapped set which is hyperbolic and of sufficiently small fractal dimension. Very recently, Bouclet and Mizutani [1] and the authors [39] showed the global-in-time one without loss of derivative also with a hyperbolic trapped geodesic in sense of [2].

On the flat cone $C(\mathbb{S}_\rho^1)$ of dimension two, Ford [9] proved the full range of global-in-time Strichartz estimates. For high dimension, the first author [37] obtained a global-in-time Strichartz estimate from the restriction estimate but with a loss of angular derivatives. On the other hand, the perturbation of the inverse-square potential is non-trivial since the inverse-square decay of the potential is in some sense critical for the spectral and scattering theory. In [3, 4], they generalized the Euclidean standard Strichartz estimate for Schrödinger and wave to the case in which an additional inverse-square potentials is present as a perturbation. The first main purpose of this paper is to prove the full range of global-in-time Strichartz estimates for Schrödinger equation associated with the operator \mathcal{L}_V , which is on the metric cone and with an inverse-square potential.

More precisely we prove the following results.

Theorem 1.1 (Global-in-time Strichartz estimate). *Suppose that (X, g) is a metric cone of dimension $n \geq 3$. Let $\mathcal{L}_V = \Delta_g + V$ where $r^2 V =: V_0 \in \mathcal{C}^\infty(Y)$ such that*

$\Delta_h + V_0(y) + (n-2)^2/4$ is a strictly positive operator on $L^2(Y)$. Then the homogenous Strichartz estimate

$$(1.3) \quad \|e^{it\mathcal{L}_V} u_0\|_{L_t^q L_z^r(\mathbb{R} \times X)} \leq C \|u_0\|_{L^2(X)},$$

holds for the admissible pair $(q, r) \in [2, \infty]^2$ satisfies (1.2); the inhomogeneous inequality

$$(1.4) \quad \left\| \int_0^t e^{i(t-s)\mathcal{L}_V} F(s) ds \right\|_{L_t^q L_z^r(\mathbb{R} \times X)} \leq C \|F\|_{L_t^{\tilde{q}'} L_z^{\tilde{r}'}(\mathbb{R} \times X)}$$

holds for admissible pairs (q, r) , (\tilde{q}, \tilde{r}) except $q = \tilde{q} = 2$. In addition, $\|V_0\|_{L^\infty(Y)} \ll 1$, (1.4) further holds for $q = \tilde{q} = 2$ and $r = \tilde{r} = \frac{2n}{n-2}$.

Remark 1.1. The result shows the full range of global-in-time Strichartz estimate without loss of derivative except for the inhomogeneous estimate $q = \tilde{q} = 2$.

We sketch the idea and argument for the proof here. Our proof combines the methods of [19] in which we developed a micro-localized spectral measure to capturing the decay and oscillation of the Schrödinger propagator, and of [37] in which we employed the Cheeger-Taylor [5]'s method to write the propagator as a linear combination of products of the Hankel transform of the radial part and eigenfunctions of $\Delta_h + V_0(y) + (n-2)^2/4$. In our metric cone setting, to analysis the spectral measure, the method in [19] works very well far away the cone tip and Cheeger-Taylor's method easily works near the cone tip. We also consider the perturbation of the inverse-square type potential, as remarked in [19, Remark 3.7], if $V_0(y)$ takes values in the range $-(n-2)^2/4, 0)$, then one follows from [13, Corollary 1.5] that the $L^1 \rightarrow L^\infty$ norm of the propagator is at least a constant times $t^{-(\nu_0+1)}$ as $t \rightarrow \infty$, where ν_0^2 is the smallest eigenvalue of $\Delta_h + V_0 + (n-2)^2/4$. Under the above assumption on the range of V_0 , we see that $\nu_0 < (n-2)/2$. This implies that the dispersive estimate (1-12) in [19] will no longer be valid as $|t-s| \rightarrow \infty$, hence we can not obtain dispersive estimate and then use Keel-Tao's abstract method to obtain the full set of Strichartz estimate as [19] did. Therefore we can show the Strichartz estimate only for \mathcal{L}_0 without potential at the this moment. However the global-in-time Strichartz estimate for \mathcal{L}_V still can be derived from the usual Rodnianski-Schlag method [26], for example the Strichartz estimate established in [4] for negative inverse-square potentials on \mathbb{R}^n . Having the full range of Strichartz estimate for \mathcal{L}_0 in hand, another key thing is a global-in-time local smoothing estimate for \mathcal{L}_V . We remark here that we closely follow [19] to establish the full range of Strichartz estimate for \mathcal{L}_0 but we need carefully consider the spectral measure near the cone tip. To establish a global-in-time local smoothing estimate, the general method is to use the resolvent estimate for whole frequency which is not an easy thing to do and has its own independently interesting. We here provide a more direct argument going around establishing the resolvent estimate for our operator \mathcal{L}_V , this is because the required global-in-time local smoothing estimate is a estimate a L^2 -based weight space, hence the formulas with separating variables expression can be directly used to prove the local smoothing inequality, though the estimates for general L^p will give rise to a loss of angular derivative as shown [37].

As an application of the global-in-time Strichartz estimates, we study the cubic Schrödinger equation

$$(1.5) \quad \begin{cases} i\partial_t u + \mathcal{L}_V u + \gamma|u|^2 u = 0, & (t, z) \in \mathbb{R} \times X, \\ u(t, z)|_{t=0} = u_0(z), & z \in X. \end{cases}$$

where $\gamma = \pm 1$ which corresponds to the defocusing and focusing case respectively. Here we consider global existence and scattering for the cubic initial value problem. In particular the dimension of the metric cone $n = 2$, we [37] obtained the global solution and scattering result for the mass-critical (1.5) with small L^2 -norm radial data. Due to the Strichartz estimate in Theorem 1.1, one can follow the arguments of Cazenave-Weissler [8] or Tao [34] to obtain the similar result for the high dimension mass-critical equation without the radial assumption. We here consider the well-posedness problem for (1.5) in energy space and we will meet a new difficulty caused by chain rule associated with our operator \mathcal{L}_V . Before stating the second result, we need some notation.

Let (X, g) be the metric manifold above and let $dv = \sqrt{g}dz = r^{n-1}drdh$ be the measure induced by the metric g , we define the complex Hilbert space $L^2(X)$ is given by the inner product

$$\langle f, h \rangle_{L^2(X)} = \int_X f(z) \overline{h(z)} dv.$$

We say that $f \in L^p(X)$ for $1 < p < \infty$ if $\int_X |f|^p dv < \infty$. For $1 \leq p < \infty$ we denote the inhomogeneous Sobolev space over X by $H_p^1(X) = (\text{Id} + \mathcal{L}_V)^{-\frac{1}{2}} L^p(X)$ and write $H^1(X) = H_p^1(X)$ with $p = 2$.

In this paper, as mentioned above, we are interested in the global existence and scattering for nonlinear equation (1.5) with $u_0 \in H^1(X)$ when the dimension $n = 3$. The initial value problem falls into a class of energy-subcritical one on the metric cone. Solutions to (1.5) preserve the energy,

$$(1.6) \quad E(u)(t) = \int_X \left(\frac{1}{2} |\sqrt{\mathcal{L}_V} u(t, z)|^2 + \frac{\gamma}{4} |u(t, z)|^4 \right) dv$$

along with the mass

$$(1.7) \quad M(u)(t) = \int_X |u(t, z)|^2 dv.$$

Our second main result about the cubic Schrödinger equation is the following

Theorem 1.2. *Let X be metric cone of dimension $n = 3$ and $\mathcal{L}_V = \Delta_g + V$ as in Theorem 1.1 and $\gamma = \pm 1$ and suppose that the initial data $u_0 \in H^1(X)$. Then there exists $T = T(\|u_0\|_{H^1}) > 0$ such that the nonlinear Schrödinger equation (1.5) has a unique solution u satisfying*

$$(1.8) \quad u \in C(I; H^1(X)) \cap L_t^q(I; H_r^1(X)), \quad I = [0, T).$$

The solution for the defocusing case, i.e. $\gamma = 1$, can be extended to a global one. Moreover assume $\|u_0\|_{H^1(X)} \leq \epsilon$ for a small constant ϵ , there exists a global solution u

and the solution u scatters in sense that there are $u_{\pm} \in H^1(X)$ such that

$$\lim_{t \rightarrow \pm\infty} \|u(t) - e^{it\mathcal{L}_V} u_{\pm}\|_{H^1(X)} = 0.$$

This theorem is an analogue of the well known result for nonlinear Schrödinger on Euclidean space and the result about the global well posedness and scattering theory for small data still holds on the metric cone manifold. We are known that the key things of the proof are the global-in-time Strichartz estimate and Leibniz chain rule in the establishment of well-posedness and scattering theory for the small initial data problem. As far as we known, there is no result about the chain rule for the operator \mathcal{L}_V which is a bit different from the classical one due to the perturbation of the inverse-square potential. For example consider the special operator \mathcal{L}_V on Euclidean space and $V = a|z|^{-2}$, Killip, Miao, Visian and the authors [21] proved the fraction chain rule whose range of the index p is restricted by the value of a . In our situation, the related results are relevant to the smallest eigenvalue of the operator $\Delta_h + V_0(y) + (n-2)^2/4$, therefore we have to choose the admissible pairs adapt to the smallest eigenvalue ν_0^2 in the argument of fixed point argument. Since we consider the Cauchy problem in energy space $H^1(X)$, the result on the boundedness of Riesz transform in Hassell-Lin [16] and the dual argument (see [27, 38]) are enough to give a chain rule required in proving our result. In the process, we apply the asymptotical behavior of resolvent in [16] to prove a Hardy inequality associated with the operator \mathcal{L}_V which has its own interesting. Due to the independently interesting of the fraction chain rule associated with \mathcal{L}_V (including the boundedness of a generalized Riesz transform operator), we will discuss more in a forthcoming paper [40].

Now we introduce some notation. We use $A \lesssim B$ to denote $A \leq CB$ for some large constant C which may vary from line to line and depend on various parameters, and similarly we use $A \ll B$ to denote $A \leq C^{-1}B$. We employ $A \sim B$ when $A \lesssim B \lesssim A$. If the constant C depends on a special parameter other than the above, we shall denote it explicitly by subscripts. For instance, C_{ϵ} should be understood as a positive constant not only depending on p, q, n , and M , but also on ϵ . Throughout this paper, pairs of conjugate indices are written as p, p' , where $\frac{1}{p} + \frac{1}{p'} = 1$ with $1 \leq p \leq \infty$. We denote a_{\pm} to be any quantity of the form $a \pm \epsilon$ for any small $\epsilon > 0$.

This paper is organized as follows: In Section 2, we use the Hankel transform and Bessel function to give the expression of the solution, and we also establish the L^p -product chain rule. Section 3 is devoted to considering the spectral measure associated with the operator $\mathcal{L}_0 = \Delta_g$ on the metric cone and we prove the Strichartz estimate for Schrödinger without perturbation of the potential. In Section 4, we prove a local-smoothing estimate and then obtain Theorem 1.1. In the final section, we use the Strichartz estimates and L^p -product chain rule to show Theorem 1.2.

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2. PRELIMINARY: ANALYSIS RESULTS ASSOCIATED WITH \mathcal{L}_V

In this section, we study the operator \mathcal{L}_V over metric cone X . By following the separation of variable method [5], we first introduce a orthogonal decomposition of $L^2(Y)$ associated with the eigenfunctions of $\Delta_h + V_0(y) + (n-2)^2/4$ and we next write the Schrödinger propagator as a linear combination of products of the Hankel transform of the radial part and eigenfunctions of $\Delta_h + V_0(y) + (n-2)^2/4$. We finally close this section by studying the Sobolev space and the L^p -product chain rule which will serve the existence theory of nonlinear solution.

2.1. Hankel transform and Bessel function. Let $(r, y) \in \mathbb{R}_+ \times Y$ be some polar coordinates, recall the operator

$$(2.1) \quad \mathcal{L}_V = \Delta_g + \frac{V_0(y)}{r^2},$$

on the metric cone $X = (0, \infty)_r \times Y$ where $V_0(y)$ is a real continuous function and the metric g in coordinates $(r, y) \in \mathbb{R}_+ \times Y$ is a metric of the form

$$g = dr^2 + r^2 h(y, dy).$$

We remark that the Riemannian metric h on Y is independent of r hence we can use the separation of variable method. We will use the Dirichelet condition for \mathcal{L}_V when Y has a boundary. Let Δ_h be the Laplace-Beltrami operator on (Y, h) , we assume that V_0 is a smooth function on Y such that

$$(2.2) \quad \Delta_h + V_0(y) + (n-2)^2/4 > 0$$

is strictly positive on $L^2(Y)$ in sense that for any $f \in L^2(Y) \setminus \{0\}$

$$\langle (\Delta_h + V_0(y) + (n-2)^2/4)f, f \rangle_{L^2(Y)} > 0.$$

We denote the smallest eigenvalue of $\Delta_h + V_0(y) + (n-2)^2/4$ by ν_0^2 and second lowest eigenvalue ν_1^2 , with $\nu_0, \nu_1 > 0$. Then $\mathcal{L}_V > 0$ in $L^2(X; dg(z))$ with $dg(z) = \sqrt{|g|}dz = r^{n-1}drdh$. Let χ_∞ be the set

$$(2.3) \quad \chi_\infty = \left\{ \nu : \nu = \sqrt{(n-2)^2/4 + \lambda}; \lambda \text{ is eigenvalue of } \Delta_h + V_0(y) \right\}.$$

For $\nu \in \chi_\infty$, let $d(\nu)$ be the multiplicity of $\lambda_\nu = \nu^2 - \frac{1}{4}(n-2)^2$ as eigenvalue of $\tilde{\Delta}_h := \Delta_h + V_0(y)$. Let $\{\varphi_{\nu, \ell}(y)\}_{1 \leq \ell \leq d(\nu)}$ be the eigenfunctions of $\tilde{\Delta}_h$, that is

$$(2.4) \quad \tilde{\Delta}_h \varphi_{\nu, \ell} = \lambda_\nu \varphi_{\nu, \ell}, \quad \langle \varphi_{\nu, \ell}, \varphi_{\nu, \ell'} \rangle_{L^2(Y)} = \delta_{\ell, \ell'} = \begin{cases} 1, & \ell = \ell' \\ 0, & \ell \neq \ell'. \end{cases}$$

Let $\mathcal{H}^\nu = \text{span}\{\varphi_{\nu, 1}, \dots, \varphi_{\nu, d(\nu)}\}$, we have the orthogonally decomposition of the $L^2(Y)$ in sense that

$$L^2(Y) = \bigoplus_{\nu \in \chi_\infty} \mathcal{H}^\nu$$

Define the orthogonal projection π_ν on $f \in L^2(X)$

$$\pi_\nu f = \sum_{\ell=1}^{d(\nu)} \varphi_{\nu,\ell}(y) \int_Y f(r, y) \varphi_{\nu,\ell}(y) dh := \sum_{\ell=1}^{d(\nu)} \varphi_{\nu,\ell}(y) a_{\nu,\ell}(r)$$

where dh is the measure on Y under the metric h . For any $f \in L^2(X)$, we can write f in the form of separation of variable

$$(2.5) \quad f(z) = \sum_{\nu \in \chi_\infty} \pi_\nu f = \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} a_{\nu,\ell}(r) \varphi_{\nu,\ell}(y)$$

and furthermore

$$(2.6) \quad \|f(z)\|_{L^2(Y)}^2 = \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} |a_{\nu,\ell}(r)|^2.$$

We write \mathcal{L}_V in the coordinate (r, y) as

$$(2.7) \quad \mathcal{L}_V = -\partial_r^2 - \frac{n-1}{r} \partial_r + \frac{1}{r^2} (\Delta_h + V_0(y)).$$

Let $\nu > -\frac{1}{2}$ and $r > 0$ and define the Bessel function of order ν by

$$J_\nu(r) = \frac{(r/2)^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(1/2)} \int_{-1}^1 e^{isr} (1-s^2)^{(2\nu-1)/2} ds.$$

A simple computation gives the rough estimates

$$(2.8) \quad |J_\nu(r)| \leq \frac{Cr^\nu}{2^\nu \Gamma(\nu + \frac{1}{2}) \Gamma(1/2)} \left(1 + \frac{1}{\nu + 1/2} \right),$$

where C is a absolute constant. Let $f \in L^2(X)$, using the Bessel function, we define the Hankel transform of order ν by

$$(2.9) \quad (\mathcal{H}_\nu f)(\rho, y) = \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) f(r, y) r^{n-1} dr.$$

Briefly recalling functional calculus for cones [32], for well-behaved functions F , we have by [32, (8.45)]

$$(2.10) \quad F(\mathcal{L}_V)g(r, y) = \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \varphi_{\nu,\ell}(y) \int_0^\infty F(\rho^2) (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) b_{\nu,\ell}(\rho) \rho^{n-1} d\rho$$

where $b_{\nu,\ell}(\rho) = (\mathcal{H}_\nu a_{\nu,\ell})(\rho)$ with $g(r, y) = \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} a_{\nu,\ell}(r) \varphi_{\nu,\ell}(y)$. For $u_0 \in L^2(X)$, we write it in the form of separation of variables by (2.5)

$$u_0(z) = \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} a_{\nu,\ell}(r) \varphi_{\nu,\ell}(y),$$

therefore the solution of the Cauchy problem

$$(2.11) \quad \begin{cases} i\partial_t u + \mathcal{L}_V u = 0, \\ u(0, z) = u_0(z), \end{cases}$$

can be written in a form of Hankel transform representation with separation of variable, by using (2.10) with $F(\rho^2) = e^{it\rho^2}$

$$\begin{aligned}
 (2.12) \quad u(t, z) &= e^{it\mathcal{L}^V} u_0 = v(t, r, y) \\
 &= \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \varphi_{\nu, \ell}(y) \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) e^{it\rho^2} b_{\nu, \ell}(\rho) \rho^{n-1} d\rho \\
 &= \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \varphi_{\nu, \ell}(y) \mathcal{H}_\nu[e^{it\rho^2} b_{\nu, \ell}(\rho)](r).
 \end{aligned}$$

where $b_{\nu, \ell}(\rho) = (\mathcal{H}_\nu a_{\nu, \ell})(\rho)$. We refer alternatively the reader to [37] for more details about this.

Next, we recall the properties of Bessel function $J_\nu(r)$ in [30, 36], and we record here the properties which is needed for our purpose as the following

Lemma 2.1 (Asymptotics of the Bessel function). *Assume $\nu \gg 1$. Let $J_\nu(r)$ be the Bessel function of order ν defined as above. Then there exist a large constant C and a small constant c independent of ν and r such that:*

- when $r \leq \frac{\nu}{2}$

$$(2.13) \quad |J_\nu(r)| \leq C e^{-c(\nu+r)};$$

- when $\frac{\nu}{2} \leq r \leq 2\nu$

$$(2.14) \quad |J_\nu(r)| \leq C \nu^{-\frac{1}{3}} (\nu^{-\frac{1}{3}} |r - \nu| + 1)^{-\frac{1}{4}};$$

- when $r \geq 2\nu$

$$(2.15) \quad J_\nu(r) = r^{-\frac{1}{2}} \sum_{\pm} a_{\pm}(r, \nu) e^{\pm i r} + E(r, \nu),$$

where $|a_{\pm}(r, \nu)| \leq C$ and $|E(r, \nu)| \leq C r^{-1}$.

As a direct consequence, we have [22, Lemma 2.2]

Lemma 2.2. *Let $R \gg 1$, then there exists a constant C independent of ν and R such that*

$$(2.16) \quad \int_R^{2R} |J_\nu(r)|^2 dr \leq C.$$

2.2. L^p -product chain rule. The L^p -product rule for fractional derivatives in Euclidean spaces was first proved by Christ and Weinstein [8]. The chain rules for differential operators of non-integer order. For example, if $1 < p < \infty$ and $s > 0$, then

$$\|(-\Delta_{\mathbb{R}^n})^{\frac{s}{2}}(uv)\|_{L^p(\mathbb{R}^n)} \lesssim \|(-\Delta_{\mathbb{R}^n})^{\frac{s}{2}}u\|_{L^{p_1}(\mathbb{R}^n)} \|v\|_{L^{p_2}(\mathbb{R}^d)} + \|u\|_{L^{p_3}(\mathbb{R}^n)} \|(-\Delta_{\mathbb{R}^n})^{\frac{s}{2}}v\|_{L^{p_4}(\mathbb{R}^n)}$$

whenever $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. For a textbook presentation of these theorems and original references, see [33]. However, there is a bit difference from this if one considers the perturbation of inverse-square potential; we refer the reader to [21]. Since we here consider the solution of the nonlinear equation on $H^1(X)$, the result corresponding to

Riesz transform seems a bit easy to be obtained due to Hassell-Lin [16] which concluded that: let $\nabla_g = (\partial_r, r^{-1}\nabla_h)$ the gradient on X , then there exists a constant C such that

$$(2.17) \quad \|\nabla_g f\|_{L^p(X)} \leq C \|\sqrt{\mathcal{L}_V} f\|_{L^p(X)};$$

if and only if p is in the interval

$$(2.18) \quad R_p := \left(\frac{n}{\min\{1 + \frac{n}{2} + \nu_0, n\}}, \frac{n}{\max\{\frac{n}{2} - \nu_0, 0\}} \right)$$

where $\nu_0 > 0$ and ν_0^2 is the smallest eigenvalue of the $\Delta_h + V_0 + (n-2)^2/4$. We record the chain rule which is enough for our application even not optimal in the range of the index.

Proposition 2.1. *Let \mathcal{L}_V as above and let $\nu_0 > 0$ such that ν_0^2 is the smallest eigenvalue of the $\Delta_h + V_0 + (n-2)^2/4$. Then for all $u, v \in \mathcal{C}_c^\infty(X^\circ)$, compactly supported smooth functions on the interior of the metric cone, we have*

$$(2.19) \quad \|\sqrt{\mathcal{L}_V}(uv)\|_{L^p(X)} \lesssim \|\sqrt{\mathcal{L}_V}u\|_{L^{p_1}(X)} \|v\|_{L^{p_2}(X)} + \|u\|_{L^{q_1}(X)} \|\sqrt{\mathcal{L}_V}v\|_{L^{q_2}(X)},$$

for any exponents satisfying $p, p', p_1, q_2 \in R_p$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$.

Remark 2.1. *The result is not optimal due to the restriction of $p' \in R_p$. In particular $n = 3$, we have $R_p = (\frac{6}{\min\{5+2\nu_0, 6\}}, \frac{6}{\max\{3-2\nu_0, 0\}})$.*

Proof. The proof is based on the boundness of Riesz transform in [16] and a dual argument in [27]. Let $f = uv$. Since $\sqrt{\mathcal{L}_V}\mathcal{C}_0^\infty$ is dense in $L^{p'}$ (see [27, Appendix]), then

$$\|\sqrt{\mathcal{L}_V}f\|_{L^p} = \sup \left\{ \langle \sqrt{\mathcal{L}_V}f, \sqrt{\mathcal{L}_V}h \rangle : h \in \mathcal{C}_0^\infty(X^\circ), \|\sqrt{\mathcal{L}_V}h\|_{L^{p'}} \leq 1 \right\}.$$

Therefore by the definition of the square root of $\mathcal{L}_V = \nabla_g^* \nabla_g + V(z)$, we see

$$\begin{aligned} \|\sqrt{\mathcal{L}_V}f\|_{L^p} &\leq \sup_{\|\sqrt{\mathcal{L}_V}h\|_{L^{p'}} \leq 1} |\langle \mathcal{L}_V^{\frac{1}{2}}f, \mathcal{L}_V^{\frac{1}{2}}h \rangle| = |\langle \mathcal{L}_V f, h \rangle| \\ &\leq \|\nabla_g f\|_{L^p} \|\nabla_g h\|_{L^{p'}} + \|V_0\|_{L^\infty(Y)} \|f/|z|\|_{L^p} \|h/|z|\|_{L^{p'}}. \end{aligned}$$

If $p, p' \in R_p$, the Hardy inequality (2.22) below implies

$$(2.20) \quad \|\sqrt{\mathcal{L}_V}f\|_{L^p} \leq C \|\nabla_g f\|_{L^p} \|\nabla_g h\|_{L^{p'}}.$$

Note that for any X_i is the vector field on the manifold X , then

$$X_i(uv) = X_i(u)v + uX_i(v).$$

For each vector field $\partial_r, r^{-1}\nabla_h$, it works and thus the chain rule works for ∇_g . Hence we prove by the chain rule for ∇_g and the Hölder inequality

$$(2.21) \quad \begin{aligned} \|\sqrt{\mathcal{L}_V}(uv)\|_{L^p} &\leq C \|\nabla_g(uv)\|_{L^p} \\ &\leq C (\|\nabla_g u\|_{L^{p_1}(X)} \|v\|_{L^{p_2}(X)} + \|u\|_{L^{q_1}(X)} \|\nabla_g v\|_{L^{q_2}(X)}). \end{aligned}$$

Hence by (2.17) we prove (2.19) if p_1, q_2 satisfies (2.18).

□

2.3. Hardy inequality. In this subsection, we prove the hardy inequality associated with \mathcal{L}_V by following the argument in [16] with minor modification. The hardy inequality says that

Proposition 2.2 (Hardy inequality for \mathcal{L}_V). *Let $n \geq 3$ and ν_0 be as above. Suppose $0 < s < \min\{1 + \nu_0, 2\}$, and $1 < p < \infty$. Then*

$$(2.22) \quad \left\| |z|^{-s} f(z) \right\|_{L^p(X)} \lesssim \left\| \mathcal{L}_V^{\frac{s}{2}} f \right\|_{L^p(X)}$$

holds for

$$(2.23) \quad n / \min\{1 + \frac{n}{2} + \nu_0, n\} < p < n / \max\{s + \frac{n}{2} - 1 - \nu_0, 0\}.$$

Remark 2.2. Note that $\nu_0 > 0$, we can choose $s = 1$, then (2.23) is exactly the interval R_p defined in (2.18).

Proof of Proposition 2.2. The estimate (2.22) is equivalent to

$$(2.24) \quad \left\| |z|^{-s} \mathcal{L}_V^{-\frac{s}{2}} h \right\|_{L^p(X)} \lesssim \|h\|_{L^p(X)}$$

where the operator $\mathcal{L}_V^{-\frac{s}{2}}$ is defined by the Riesz potentials kernel

$$\mathcal{L}_V^{-\frac{s}{2}}(z, z') := \int_0^\infty \lambda^{1-s} (\mathcal{L}_V + \lambda^2)^{-1}(z, z') d\lambda.$$

Therefore we consider the operator $T = |z|^{-s} \mathcal{L}_V^{-\frac{s}{2}}$ with its kernel

$$T(z, z') = |z|^{-s} \mathcal{L}_V^{-\frac{s}{2}}(z, z').$$

Following the method [16], we study the kernel $T(z, z')$ based on the resolvent kernel $(\mathcal{L}_V + \lambda^2)^{-1}(z, z')$.

Step 1: L^2 -bounded. Note $dv = \sqrt{|g|} dz = r^{n-1} dr dh$. In particular $p = 2$, we compute that by (2.5) and (2.6)

$$\begin{aligned} & \left\| |z|^{-1} f(z) \right\|_{L^2(X)}^2 \\ &= \int_0^\infty \int_Y \frac{|\sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} a_{\nu, \ell}(r) \varphi_{\nu, \ell}(y)|^2}{r^2} r^{n-1} dr dh \\ &= \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \int_0^\infty \frac{|a_{\nu, \ell}(r)|^2}{r^2} r^{n-1} dr \\ &\lesssim \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \|\partial_r a_{\nu, \ell}(r)\|_{L^2(r^{n-2} dr)}^2 \lesssim \|\nabla_g f\|_{L^2(X)}^2 \lesssim \|\mathcal{L}_V^{1/2} f\|_{L^2(X)}^2. \end{aligned}$$

Therefore T is bounded on $L^2(X)$.

Step 2: L^p -bounded on far away diagonal region. Let $\chi [0, \infty) \rightarrow [0, 1]$ be a smooth cutoff function such that $\chi([0, 1/2]) = 1$ and $\chi([1, \infty)) = 0$, define the operators

$$(2.25) \quad T_1(z, z') = \chi(4r/r') r^{-s} \int_0^\infty \lambda^{1-s} (\mathcal{L}_V + \lambda^2)^{-1}(z, z') d\lambda;$$

$$(2.26) \quad T_2(z, z') = \chi(4r'/r) r^{-s} \int_0^\infty \lambda^{1-s} (\mathcal{L}_V + \lambda^2)^{-1}(z, z') d\lambda.$$

Then we decompose $T = T_0 + T_1 + T_2$ where

$$(2.27) \quad T_0(z, z') = (1 - \chi(4r/r') - \chi(4r'/r)) r^{-s} \int_0^\infty \lambda^{1-s} (\mathcal{L}_V + \lambda^2)^{-1}(z, z') d\lambda.$$

In this subsection, we consider the L^p -boundedness of T_1 and T_2 . Since \mathcal{L}_V is homogeneous of degree -2 , then we have by scaling

$$(\mathcal{L}_V + \lambda^2)^{-1}(z, z') = \lambda^{n-2} (\mathcal{L}_V + 1)^{-1}(\lambda z, \lambda z').$$

To this end, we need two results from [16]. One is about the resolvent kernel $(\mathcal{L}_V + 1)^{-1}(z, z')$ in [16, Theorem 4.11] and the other is about the boundedness of a kind of operator in [16, Corollary 5.9]. By [16, Theorem 4.11], we have

$$(2.28) \quad |\chi(4r/r')(\mathcal{L}_V + 1)^{-1}(z, z')| \lesssim r^{1-\frac{n}{2}+\nu_0} r'^{1-\frac{n}{2}-\nu_0} \langle r' \rangle^{-\infty}.$$

and

$$(2.29) \quad |\chi(4r'/r)(\mathcal{L}_V + 1)^{-1}(z, z')| \lesssim r^{1-\frac{n}{2}-\nu_0} r'^{1-\frac{n}{2}+\nu_0} \langle r \rangle^{-\infty}.$$

Therefore we have that by (2.28) for any $N > 1 - s$ and $s < 2$

$$(2.30) \quad \begin{aligned} |T_1(z, z')| &\lesssim \left| r^{-s} \int_0^\infty \lambda^{n-1-s} \chi(4r/r') (\mathcal{L}_V + 1)^{-1}(\lambda z, \lambda z') d\lambda \right| \\ &\lesssim r^{-s} r'^{2-n} (r/r')^{1-\frac{n}{2}+\nu_0} \left(\int_0^{1/r'} \lambda^{1-s} d\lambda + r'^{-N} \int_{1/r'}^\infty \lambda^{1-s-N} d\lambda \right) \\ &\lesssim r'^{-n} (r/r')^{1-\frac{n}{2}+\nu_0-s}. \end{aligned}$$

Similarly we have

$$(2.31) \quad \begin{aligned} |T_2(z, z')| &\lesssim \left| r^{-s} \int_0^\infty \lambda^{n-1-s} \chi(4r'/r) (\mathcal{L}_V + 1)^{-1}(\lambda z, \lambda z') d\lambda \right| \\ &\lesssim r^{-s} r^{2-n} (r'/r)^{1-\frac{n}{2}+\nu_0} \left(\int_0^{1/r} \lambda^{1-s} d\lambda + r^{-N} \int_{1/r}^\infty \lambda^{1-s-N} d\lambda \right) \\ &\lesssim r^{-n} (r'/r)^{1-\frac{n}{2}+\nu_0-s}. \end{aligned}$$

Hence by [16, Corollary 5.9], we have that T_1 is bounded on $L^p(X)$ for $1 < p < n/\max\{s + \frac{n}{2} - 1 - \nu_0, 0\}$ and T_2 is bounded on $L^p(X)$ for $p > n/\min\{1 + \frac{n}{2} + \nu_0, n\}$.

Step 3: L^p -bounded on diagonal region. Recall the distance on a metric cone is

$$(2.32) \quad d(z, z') = \begin{cases} \sqrt{r^2 + r'^2 - 2rr' \cos(d_Y(y, y'))}, & d_Y(y, y') \leq \pi; \\ r + r', & d_Y(y, y') \geq \pi. \end{cases}$$

In the diagonal region, i.e. the support of $1 - \chi(4r/r') - \chi(4r'/r)$, we have

$$(2.33) \quad d(z, z')^{-1} \geq (r + r')^{-1} = r^{-1}(1 + r'/r)^{-1} \geq r^{-1}/9.$$

We claim the following conclusions:

- (i) T_0 is bounded on $L^2(X)$;
- (ii) $|T_0(z, z')| \leq Cd(z, z')^{-n}$;
- (iii) $|\nabla_z T_0(z, z')| \leq Cd(z, z')^{-(n+1)}$ and $|\nabla_{z'} T_0(z, z')| \leq Cd(z, z')^{-(n+1)}$.

Now we verify these conclusions. We have seen that T, T_1, T_2 are bounded on $L^2(X)$ from the first two steps if $s < 1 + \nu_0$, hence $T_0 = T - T_1 - T_2$ is also bounded on $L^2(X)$. To verify (ii) and (iii), we recall [16, Lemma 5.4] which implies for any integer $j \geq 0$ and any $N > 0$

$$\left| \nabla_{z, z'}^j K(z, z') \right| \lesssim \begin{cases} d(z, z')^{2-n-j}, & d(z, z') \leq 1; \\ d(z, z')^{-N}, & d(z, z') \geq 1. \end{cases}$$

where $K(z, z') = (1 - \chi(4r/r') - \chi(4r'/r))(\mathcal{L}_V + 1)^{-1}(z, z')$. Therefore we compute that by using $d(\lambda z, \lambda z') = \lambda d(z, z')$

$$|K(\lambda z, \lambda z')| \lesssim \begin{cases} \lambda^{2-n} d(z, z')^{2-n}, & d(z, z') \leq 1/\lambda; \\ \lambda^{-N} d(z, z')^{-N}, & d(z, z') \geq 1/\lambda. \end{cases}$$

and

$$|\nabla_{z, z'}(K(\lambda z, \lambda z'))| \lesssim \begin{cases} \lambda^{2-n} d(z, z')^{1-n}, & d(z, z') \leq 1/\lambda; \\ \lambda^{-N+1} d(z, z')^{-N}, & d(z, z') \geq 1/\lambda. \end{cases}$$

Note that (2.33), we finally verify (ii) by estimating for $s < 2$ and $N > n - s$

$$\begin{aligned} |T_0(z, z')| &= r^{-s} \int_0^\infty \lambda^{n-1-s} |K(\lambda z, \lambda z')| d\lambda \\ &\lesssim d(z, z')^{-s} \left(d(z, z')^{2-n} \int_0^{1/d(z, z')} \lambda^{1-s} d\lambda + d(z, z')^{-N} \int_{1/d(z, z')}^\infty \lambda^{n-1-s-N} d\lambda \right) \\ &\lesssim d(z, z')^{-n}. \end{aligned}$$

Similarly we have

$$\begin{aligned} |\nabla_z T_0(z, z')| &\lesssim r^{-s-1} \int_0^\infty \lambda^{n-1-s} |K(\lambda z, \lambda z')| d\lambda + r^{-s} \int_0^\infty \lambda^{n-1-s} |\nabla_z(K(\lambda z, \lambda z'))| d\lambda \\ &\lesssim d(z, z')^{-n-1} + d(z, z')^{-s} \left(d(z, z')^{1-n} \int_0^{1/d(z, z')} \lambda^{1-s} d\lambda + d(z, z')^{-N} \int_{1/d(z, z')}^\infty \lambda^{n-s-N} d\lambda \right) \\ &\lesssim d(z, z')^{-n-1}. \end{aligned}$$

We also have $|\nabla_{z'} T_0(z, z')| \lesssim d(z, z')^{-n-1}$, thus we prove (iii). As a consequence of Calderón-Zygmund theory, we have T_0 is bounded from $L^1(X)$ to weak $L^1(X)$. By using interpolation, we obtain T_0 is bounded on $L^p(X)$ for all $1 < p \leq 2$. By dual, we show the following result

Proposition 2.3. *The operator T_0 is a bounded operator on $L^p(X)$ for all $p > 1$.*

Collecting all results from the last two steps, we prove T is bounded on $L^p(X)$ for all p satisfies (2.23). Therefore we prove Proposition 2.2. \square

3. SPECTRAL MEASURE ASSOCIATED WITH \mathcal{L}_0

In this section, we consider the spectral measure associated with the operator $\mathcal{L}_0 = \Delta_g$ on the metric cone. In the slightly different setting of asymptotically conic manifolds, Hassell and the first author [19, Proposition 1.5] have showed

Proposition 3.1. *Let (M°, g) be non-trapping asymptotically conic manifold and $\mathbf{H} = \Delta_g$. Then there exists a λ -dependent operator partition of unity on $L^2(M^\circ)$*

$$\text{Id} = \sum_{j=1}^N Q_j(\lambda),$$

with N independent of λ , such that for each $1 \leq j \leq N$ we can write

$$(3.1) \quad (Q_j(\lambda) dE_{\sqrt{\mathbf{H}}}(\lambda) Q_j^*(\lambda))(z, z') = \lambda^{n-1} \left(\sum_{\pm} e^{\pm i \lambda d(z, z')} a_{\pm}(\lambda, z, z') + b(\lambda, z, z') \right),$$

with estimates

$$(3.2) \quad |\partial_{\lambda}^{\alpha} a_{\pm}(\lambda, z, z')| \leq C_{\alpha} \lambda^{-\alpha} (1 + \lambda d(z, z'))^{-\frac{n-1}{2}},$$

$$(3.3) \quad |\partial_{\lambda}^{\alpha} b(\lambda, z, z')| \leq C_{\alpha, M} \lambda^{-\alpha} (1 + \lambda d(z, z'))^{-K} \text{ for any } K.$$

Here $d(\cdot, \cdot)$ is the Riemannian distance on M° .

Once the analogue of spectral measure associated with the operator \mathcal{L}_0 has been established, we can follow the argument in [19] to prove the Strichartz inequality. Before doing this, we gives an explicit formula for the spectral measure.

Lemma 3.1. *Let ν_j^2 be the eigenvalues of the positive operator $\Delta_h + V_0(y) + (n-2)^2/4$ and let $\varphi_j(y)$ be L^2 -normalized corresponding eigenfunction, we have the explicit formula for the spectral measure*

$$(3.4) \quad dE_{\sqrt{\mathcal{L}_V}}(1, z, z') = \frac{\pi}{2} (rr')^{-\frac{n-2}{2}} \sum_j \varphi_j(y) \overline{\varphi_j(y')} J_{\nu_j}(r) J_{\nu_j}(r').$$

Remark 3.1. *This expression provides little asymptotic behavior of the kernel, as both r, r' go to ∞ , but gives good converges as both r, r' go to 0.*

Proof. The proof is from [12, 16] which gives a explicit formula for the resolvent $(\mathcal{L}_V - (1 + i0))^{-1}$. Write the the operator \mathcal{L}_V on X as

$$(3.5) \quad \begin{aligned} \mathcal{L}_V &= -\partial_r^2 - \frac{n-1}{r} \partial_r + \frac{\Delta_h}{r^2} + \frac{V_0(y)}{r^2} \\ &= r^{-1-\frac{n}{2}} \left(-(r\partial_r)^2 + \Delta_h + V_0(y) + (n-2)^2/4 \right) r^{\frac{n}{2}-1} \\ &:= r^{-1-\frac{n}{2}} P_b r^{\frac{n}{2}-1} \end{aligned}$$

where

$$P_b = -(r\partial_r)^2 + \Delta_h + V_0(y) + (n-2)^2/4.$$

Regard naturally this operators as acting on half-densities by the formula

$$\mathcal{L}_V(f|dg|^{\frac{1}{2}}) = (\mathcal{L}_V f)|dg|^{\frac{1}{2}}, \quad \text{with } dg = r^{n-1}drdh,$$

that means the derivatives are endowed with the flat connection on half-densities that annihilates the scattering Riemannian half-density $|dg|^{\frac{1}{2}}$.

If denote the Schwartz kernel of $(P_b + k^2 r^2)^{-1}$ by G , then the Schwartz kernel of $(\mathcal{L}_V + k^2)^{-1}$ is $rr'G$ with respect to b-half density

$$w = |dg_b dg'_b|^{\frac{1}{2}} = \left| \frac{dr}{r} \frac{dr'}{r'} dy dy' \right|^{\frac{1}{2}}, \quad \text{with } dg_b = r^{-1}drdh.$$

Now consider the resolvent kernel

$$(3.6) \quad G = (P_b + k^2 r^2)^{-1} \quad \text{for } k > 0.$$

Recall ν_j^2 is the eigenvalues of the positive operator

$$\Delta_h + V_0(y) + (n-2)^2/4$$

corresponding to the L^2 -normalized eigenfunction $\varphi_j(y) = \varphi_{\nu_j}(y)$, that is

$$(\Delta_h + V_0(y) + (n-2)^2/4)\varphi_j = \nu_j^2 \varphi_j.$$

Let $\Pi_j = \pi_{\nu_j}$ be projection onto the φ_j eigen-space. Then we have the decomposition

$$P_b + k^2 r^2 = \sum_j (-(r\partial_r)^2 + k^2 r^2 + \nu_j^2) \Pi_j.$$

Therefore

$$(P_b + k^2 r^2)^{-1} = \sum_j \Pi_j (-(r\partial_r)^2 + k^2 r^2 + \nu_j^2)^{-1}.$$

To find the kernel of $(-(r\partial_r)^2 + k^2 r^2 + \nu_j^2)^{-1}$, we need to solve for $\phi_j(r)$ such that

$$(3.7) \quad (-(r\partial_r)^2 + k^2 r^2 + \nu_j^2)\phi_j(r) = \delta(r/r' - 1) = r'\delta(r - r').$$

Let $T_j = -(r\partial_r)^2 + k^2 r^2 + \nu_j^2$ and $T_j^{-1}(r, r')$ be the kernel of the inverse T_j^{-1} . When $r \neq r'$, as in [12], the solution space is spanned by the modified Bessel functions

$$(3.8) \quad \begin{aligned} I_{\nu_j}(r) &= \frac{2^{-\nu_j} r^{\nu_j}}{\sqrt{\pi} \Gamma(\nu_j + \frac{1}{2})} \int_{-1}^1 (1-t^2)^{\nu_j - \frac{1}{2}} e^{-rt} dt, \\ K_{\nu_j}(r) &= \frac{\sqrt{\pi} 2^{-\nu_j} r^{\nu_j}}{\Gamma(\nu_j + \frac{1}{2})} \int_1^\infty (t^2 - 1)^{\nu_j - \frac{1}{2}} e^{-rt} dt. \end{aligned}$$

Therefore, we have for some constants a_l, b_l with $l = 1, 2$

$$T_j^{-1}(r, r') = (a_1 I_{\nu_j}(r) + b_1 K_{\nu_j}(r))(a_2 I_{\nu_j}(r') + b_2 K_{\nu_j}(r')) \left| \frac{dr}{r} \frac{dr'}{r'} \right|^{\frac{1}{2}}, \quad r < r'.$$

Since our Laplacian here is Friedrichs extension, the derivatives of $rr'T_j^{-1}(r, r')$ to be in $L^2(X)$ which is the requirement of the domain of the Friedrichs extension. Therefore

$$(3.9) \quad T_j^{-1}(r, r') = \begin{cases} I_{\nu_j}(kr) K_{\nu_j}(kr') \left| \frac{dr}{r} \frac{dr'}{r'} \right|^{\frac{1}{2}}, & r < r'; \\ K_{\nu_j}(kr) I_{\nu_j}(kr') \left| \frac{dr}{r} \frac{dr'}{r'} \right|^{\frac{1}{2}}, & r > r'. \end{cases}$$

Hence we obtain an explicit formulae separating the r and y variables for the resolvent kernel

$$(3.10) \quad (P_b + k^2 r^2)^{-1} = \begin{cases} \sum_j \varphi_j(y) \overline{\varphi_j(y')} I_{\nu_j}(kr) K_{\nu_j}(kr') \left| \frac{dr}{r} \frac{dr'}{r'} dy dy' \right|^{\frac{1}{2}}, & r < r'; \\ \sum_j \varphi_j(y) \overline{\varphi_j(y')} K_{\nu_j}(kr) I_{\nu_j}(kr') \left| \frac{dr}{r} \frac{dr'}{r'} dy dy' \right|^{\frac{1}{2}}, & r > r'. \end{cases}$$

This formula analytically continues to the imaginary axis, so setting $k = -i$, and using the formulae

$$I_\nu(-iz) = e^{-\nu\pi i/2} J_\nu(z), \quad K_\nu(-iz) = \frac{\pi i}{2} e^{\nu\pi i/2} H_\nu^{(1)}(z),$$

we see that

$$(3.11) \quad (\mathcal{L}_V - (1 + i0))^{-1} = \begin{cases} \frac{\pi i r r'}{2} \sum_j \varphi_j(y) \overline{\varphi_j(y')} J_{\nu_j}(r) H_{\nu_j}^{(1)}(r') \left| \frac{dr dr'}{r r'} dy dy' \right|^{\frac{1}{2}}, & r < r'; \\ \frac{\pi i r r'}{2} \sum_j \varphi_j(y) \overline{\varphi_j(y')} J_{\nu_j}(r') H_{\nu_j}^{(1)}(r) \left| \frac{dr dr'}{r r'} dy dy' \right|^{\frac{1}{2}}, & r > r', \end{cases}$$

where J_ν and $H_\nu^{(1)}$ are standard Bessel and Hankel functions. Since $\text{Im}(iH_\nu^{(1)})(r) = J_\nu(r)$, let H be the Heaviside function, we have

$$\begin{aligned} dE_{\sqrt{\mathcal{L}_V}}(1, z, z') &= \text{Im} \left((\mathcal{L}_V - (1 + i0))^{-1} \right) \\ &= \frac{\pi r r'}{2} \sum_j \varphi_j(y) \overline{\varphi_j(y')} \text{Im} \left(J_{\nu_j}(r) i H_{\nu_j}^{(1)}(r') H(r' - r) + J_{\nu_j}(r') i H_{\nu_j}^{(1)}(r) H(r - r') \right) \left| \frac{dr dr'}{r r'} dy dy' \right|^{\frac{1}{2}} \\ &= \frac{\pi r r'}{2} \sum_j \varphi_j(y) \overline{\varphi_j(y')} \left(J_{\nu_j}(r) J_{\nu_j}(r') H(r' - r) + J_{\nu_j}(r') J_{\nu_j}(r) H(r - r') \right) \left| \frac{dr dr'}{r r'} dy dy' \right|^{\frac{1}{2}} \\ &= \frac{\pi r r'}{2} \sum_j \varphi_j(y) \overline{\varphi_j(y')} J_{\nu_j}(r) J_{\nu_j}(r') \left| \frac{dr dr'}{r r'} dy dy' \right|^{\frac{1}{2}}. \end{aligned}$$

Now return to half density $|dg|^{1/2}$, we obtain

$$dE_{\sqrt{\mathcal{L}_V}}(1, z, z') = \frac{\pi}{2} (r r')^{-\frac{n-2}{2}} \sum_j \varphi_j(y) \overline{\varphi_j(y')} J_{\nu_j}(r) J_{\nu_j}(r') \left| dg dg' \right|^{\frac{1}{2}}.$$

□

Now we microlocalize the spectral measure of \mathcal{L}_0 to capture the decay and oscillation behavior, as described in Proposition 3.2. According to the Stone's formulae, the Schwartz kernel of the spectral measure can be expressed in terms of the difference between the outgoing and incoming resolvents $R(\lambda \pm i0)$, where $R(\sigma) = (\mathcal{L}_0 - \sigma^2)^{-1}$. More precisely,

$$dE_{\sqrt{\mathcal{L}_0}}(\lambda) = \frac{d}{d\lambda} E_{\sqrt{\mathcal{L}_0}}(\lambda) d\lambda = \frac{\lambda}{\pi i} (R(\lambda + i0) - R(\lambda - i0)) d\lambda, \quad \lambda > 0.$$

Since \mathcal{L}_0 is homogeneous of degree -2 , then

$$(\mathcal{L}_0 - \lambda^2)^{-1}(z, z') = \lambda^{n-2} (\mathcal{L}_0 - 1)^{-1}(\lambda z, \lambda z'),$$

hence

$$dE_{\sqrt{\mathcal{L}_0}}(\lambda, z, z') = \frac{\lambda^{n-1}}{\pi i} dE_{\sqrt{\mathcal{L}_0}}(1, \lambda z, \lambda z').$$

Let $\tilde{z} = \lambda z$ and $\tilde{z}' = \lambda z'$, we consider the spectral measure $dE_{\sqrt{\mathcal{L}_0}}(1, \tilde{z}, \tilde{z}')$. Since an asymptotically conic manifold M° is defined a manifold, outside some compact set, is isometric to the conical space X far away the cone tip, hence \mathcal{L}_0 restricted on $X \cap \{|\tilde{z}| > R\}$ is same as \mathbf{H} at $M^\circ \cap \{|\tilde{z}| > R\}$ for some constant $R \gg 1$. We compactified the manifold X and M° , denoted by \bar{X} and \bar{M} respectively. We take $\tilde{x} = 1/|\tilde{z}|$ as the boundary defining function, then for any scattering pseudo-differential operator Q (denote its adjoint Q^*) micro-localized near the boundary, say $\tilde{x} < 2\epsilon = 2/R$, we have that

$$QdE_{\sqrt{\mathcal{L}_0}}(1, \tilde{z}, \tilde{z}')Q^* = QdE_{\sqrt{\mathbf{H}}}(1, \tilde{z}, \tilde{z}')Q^*.$$

Because of this, we can employ a similar partition of the identity as in [14, 19] defined by specifying the symbols of these operators, which must form a partition of unity on the phase space. Then for most Q_j which micro-localized near the boundary $\tilde{x} < 2\epsilon$, we have the property (3.16), (3.17) and (3.18). Indeed, and for a given small positive $\epsilon = 1/R$, let O_0 consist of all points away from the boundary, that is, the points with $\tilde{x} > \epsilon$. We next define O_1 to consist of points near the boundary, say $\tilde{x} < 2\epsilon$, but away from the characteristic variety, that is, satisfying $|\mu|_h^2 + \nu^2 < 1/2$ or $|\mu|_h^2 + \nu^2 > 3/2$. Finally divide the set $\{\tilde{x} < 2\epsilon, |\mu|_h^2 + \nu^2 \in [1/4, 2]\}$ into a finite number of sets O_2, \dots, O_N such that, for each set O_j , the micro-support of Q_j is contained in the set Q_j , $\{\tilde{x} < 2\epsilon, |\mu|_h^2 + \nu^2 \in [1/4, 2]; |\nu - \nu_j| \leq \delta\}$, where δ is taken sufficiently small and ν_j is the finite points in this set. Hence we define an open cover $O_0 \cup \dots \cup O_N$ of phase space.

We then form a partition of unity subordinate to the above open cover, and take these as the principal symbols of pseudo-differential operators Q_j in the class Ψ_k^0 described in [19] and are supported in O_j , ($j \geq 0$). Hence for $j \geq 1$ we have

$$Q_j dE_{\sqrt{\mathcal{L}_0}}(1, \tilde{z}, \tilde{z}')Q_j^* = Q_j dE_{\sqrt{\mathbf{H}}}(1, \tilde{z}, \tilde{z}')Q_j^* = \sum_{\pm} e^{\pm id(\tilde{z}, \tilde{z}')} a_{\pm}(1, \tilde{z}, \tilde{z}') + b(1, \tilde{z}, \tilde{z}').$$

From the proof of Proposition 1.5 in [19, section 4] and recall $\tilde{z} = \lambda z, \tilde{z}' = \lambda z'$, we see that

$$a_{\pm}(1, \tilde{z}, \tilde{z}') = a_{\pm}(\lambda, z, z'), \quad b(1, \tilde{z}, \tilde{z}') = b(\lambda, z, z').$$

On the other hand, the conic metric distant function is homogeneous $d(\tilde{z}, \tilde{z}') = \lambda d(z, z')$. Actually the conic metric d has an explicit expression $d(z, z') = r + r'$ when $d_{\partial X}(y, y') \geq \pi$ and when $d_Y(y, y') \leq \pi$

$$d(z, z') = d(y, y', r, r') = \sqrt{r^2 + r'^2 - 2rr' \cos d_Y(y, y')}$$

where we write $r = 1/x$ and $r' = 1/x'$. Therefore we have the analogue property (3.17), (3.18) for the microlocalized spectral measure when $j \geq 1$. Now we consider the spectral measure when it is micro-localize supported in $\{|\tilde{z}| \leq R\}$. From Lemma 3.1, we have

$$\begin{aligned} (3.12) \quad & Q_0 dE_{\sqrt{\mathcal{L}_0}}(1, \tilde{z}, \tilde{z}')Q_0^* \\ &= \frac{\pi}{2} (\lambda^2 r r')^{-\frac{n-2}{2}} \sum_j \varphi_j(y) \overline{\varphi_j(y')} J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda r') \chi(\lambda r) \chi(\lambda r'). \end{aligned}$$

where $\chi \in \mathcal{C}_0^\infty$ such that $\chi(s) = 1$ for $s \leq R$ and $\chi(s) = 0$ for $s \geq 2R$, for some sufficiently large R . Recall that if $\operatorname{Re} \nu > -1/2$, then

$$\frac{d}{dt} (t^{-\nu} J_\nu(t)) = -t^{-\nu} J_{\nu+1}(t)$$

and the Bessel function satisfies

$$(3.13) \quad |J_\nu(t)| \leq \frac{Ct^\nu}{2^\nu \Gamma(\nu + \frac{1}{2}) \Gamma(1/2)} \left(1 + \frac{1}{\nu + 1/2}\right).$$

We remark that we here consider $V = 0$, hence the smallest eigenvalue $\nu_0 > (n-2)/2$. Note that $\|\nabla^{(k)} \varphi_j\|_{L^\infty} \leq C_k \nu_j^{(n-1)/2+k}$, we compute that

$$(3.14) \quad \begin{aligned} & \left| \left(\frac{d}{d\lambda} \right)^\alpha \left((\lambda^2 r r')^{-\frac{n-2}{2}} \sum_j \varphi_j(y) \overline{\varphi_j(y')} J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda r') \chi(\lambda r) \chi(\lambda r') \right) \right| \\ & \lesssim \lambda^{-\alpha} (\lambda^2 r r')^{-\frac{n-2}{2}} \sum_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq \alpha} (\lambda r)^{\alpha_1 + \alpha_3} (\lambda r')^{\alpha_2 + \alpha_4} \\ & \quad \times \left| \sum_j \varphi_j(y) \overline{\varphi_j(y')} J_{\nu_j + \alpha_1}(\lambda r) J_{\nu_j + \alpha_2}(\lambda r') \chi^{(\alpha_3)}(\lambda r) \chi^{(\alpha_4)}(\lambda r') \right| \\ & \lesssim \lambda^{-\alpha} \sum_j \nu_j^{n-1} \frac{(\lambda r)^{\nu_j - \frac{n-2}{2}}}{2^{\nu_j} \Gamma(\nu_j + \frac{1}{2}) \Gamma(1/2)} \frac{(\lambda r')^{\nu_j - \frac{n-2}{2}}}{2^{\nu_j} \Gamma(\nu_j + \frac{1}{2}) \Gamma(1/2)} \chi^{(\alpha_3)}(\lambda r) \chi^{(\alpha_4)}(\lambda r') \\ & \lesssim \lambda^{-\alpha} \sum_j \nu_j^{n-1} \frac{R^{2\nu_j - (n-2)}}{2^{2\nu_j} \Gamma(\nu_j + \frac{1}{2}) \Gamma(\nu_j + \frac{1}{2})} \lesssim \lambda^{-\alpha}. \end{aligned}$$

On the other hand, since $|\tilde{z}| \leq R$ and $|\tilde{z}'| \leq R$, hence $d(\tilde{z}, \tilde{z}') \leq C(R)$. Then we have that for any $K > 0$

$$(3.15) \quad \begin{aligned} & \left(\frac{d}{d\lambda} \right)^\alpha \left((\lambda^2 r r')^{-\frac{n-2}{2}} \sum_j \varphi_j(y) \overline{\varphi_j(y')} J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda r') \chi(\lambda r) \chi(\lambda r') \right) \\ & \leq C(R) \lambda^{-\alpha} (1 + d(\tilde{z}, \tilde{z}'))^{-K} \leq \tilde{C}(R) \lambda^{-\alpha} (1 + \lambda d(z, z'))^{-K} \end{aligned}$$

which has the property (3.18) of $b(\lambda, z, z')$.

In summary, we have proved an analogue of spectral measure associated with the operator \mathcal{L}_0

Proposition 3.2. *Let (X, g) be metric cone manifold and $\mathcal{L}_0 = \Delta_g$. Then there exists a λ -dependent operator partition of unity on $L^2(X)$*

$$\operatorname{Id} = \sum_{j=0}^N Q_j(\lambda),$$

with N independent of λ , such that for each $0 \leq j \leq N$ we can write

$$(3.16) \quad (Q_j(\lambda) dE_{\sqrt{\mathcal{L}_0}}(\lambda) Q_j^*(\lambda))(z, z') = \lambda^{n-1} \left(\sum_{\pm} e^{\pm i \lambda d(z, z')} a_{\pm}(\lambda, z, z') + b(\lambda, z, z') \right),$$

with estimates

$$(3.17) \quad |\partial_\lambda^\alpha a_\pm(\lambda, z, z')| \leq C_\alpha \lambda^{-\alpha} (1 + \lambda d(z, z'))^{-\frac{n-1}{2}},$$

$$(3.18) \quad |\partial_\lambda^\alpha b(\lambda, z, z')| \leq C_{\alpha, M} \lambda^{-\alpha} (1 + \lambda d(z, z'))^{-K} \text{ for any } K.$$

Here $d(\cdot, \cdot)$ is the Riemannian distance on X .

Then we follow the argument in [19] to microlocalized (in phase space) Schrödinger propagators $U_j(t)$ associated with Q_j by

$$(3.19) \quad U_j(t) = \int_0^\infty e^{it\lambda^2} Q_j(\lambda) dE_{\sqrt{\mathcal{L}_0}}(\lambda), \quad 0 \leq j \leq N,$$

where $Q_j(\lambda)$ is a partition of the identity operator in $L^2(X^\circ)$. Then the operator $U_j(t)U_j(s)^*$ is given

$$(3.20) \quad U_j(t)U_j(s)^* = \int e^{i(t-s)\lambda^2} Q_j(\lambda) dE_{\sqrt{\mathcal{L}_0}}(\lambda) Q_j(\lambda)^*.$$

We proved a uniform estimate on $\|U_j(t)\|_{L^2 \rightarrow L^2}$ and dispersive estimate for $U_j(t)U_j(s)^*$ with norm $O(|t-s|^{-\frac{n}{2}})$, the homogeneous Strichartz estimate for $e^{it\mathcal{L}_0}$ finally was obtained by Keel-Tao's formalism [20] to each U_j and summing over j . For the endpoint inhomogeneous estimate, we required additional argument [19, section 8] to obtain dispersive estimate on $U_i(t)U_j(s)^*$ for $i \neq j$ and the Keel-Tao's argument showed the desirable endpoint inhomogeneous Strichartz estimate. As remarked in [20], we can sharp the inequality to Lorentz space norm $L^{r,2}(X)$.

Proposition 3.3. *Let $(q, r), (\tilde{q}, \tilde{r})$ be admissible pairs satisfying (1.2), the following Strichartz estimates hold: the homogeneous inequality*

$$(3.21) \quad \|e^{it\mathcal{L}_0} u_0\|_{L^q(\mathbb{R}; L^{r,2}(X))} \leq C \|u_0\|_{L^2(X)}$$

and inhomogeneous Strichartz estimate

$$(3.22) \quad \left\| \int_0^t e^{i(t-s)\mathcal{L}_0} F(s) ds \right\|_{L^q(\mathbb{R}; L^{r,2}(X))} \leq C \|F\|_{L^{\tilde{q}'}(\mathbb{R}; L^{\tilde{r}',2}(X))}.$$

Remark 3.2. *We obtain the full set of global-in-time Strichartz estimate both in homogeneous and inhomogeneous inequalities.*

4. LOCAL-SMOOTHING AND STRICHARTZ ESTIMATES

In this section, we prove a local-smoothing estimate and then obtain the Strichartz estimate by using the Rodnianski-Schlag method [26]. We remark here that we directly prove the local smoothing by a method in [22] going around the resolvent estimate of \mathcal{L}_V .

Proposition 4.1 (local-smoothing). *Let u be the solution of (1.1), then there exists a constant C independent of u_0 such that*

$$(4.1) \quad \| |z|^{-\beta} \partial_t^\alpha \mathcal{L}_V^{\frac{s}{2}} u(t, z) \|_{L_t^2(\mathbb{R}; L^2(X))} \leq C \|u_0\|_{\dot{H}^{2\alpha+s+\beta-1}(X)}$$

where $\alpha, s \in \mathbb{R}$ and $1/2 < \beta < 1 + \nu_0$ with ν_0 being the positive square root of the smallest eigenvalue of $\Delta_h + V_0(y) + (n-2)^2/4$.

Proof. Recall $dv = r^{n-1}drdh(y)$. By the Plancherel theorem with respect to time t , it suffices to estimate

$$\int_{\mathbb{R}} \int_X |\partial_t^\alpha \mathcal{L}_V^{\frac{s}{2}} u(t, z)|^2 \frac{dt dv}{|z|^{2\beta}} = \int_X \int_{\mathbb{R}} |\tau^\alpha \mathcal{L}_V^{\frac{s}{2}} \hat{u}(z, \tau)|^2 \frac{d\tau dv}{|z|^{2\beta}}.$$

where we have by (2.10) with $F(\rho^2) = \rho^s e^{it\rho^2}$

$$\begin{aligned} \mathcal{L}_V^{\frac{s}{2}} \hat{u}(\tau) &= \int_{\mathbb{R}} e^{-it\tau} \mathcal{L}_V^{\frac{s}{2}} u(z, t) dt \\ (4.2) \quad &= \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \varphi_{\nu, \ell}(y) \int_{\mathbb{R}} \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) e^{it(\rho^2 - \tau)} b_{\nu, \ell}(\rho) \rho^s \rho^{n-1} d\rho dt. \end{aligned}$$

Put this into above formula, we need to estimate

$$\begin{aligned} &\int_X \int_{\mathbb{R}} \left| \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \varphi_{\nu, \ell}(y) \int_{\mathbb{R}} \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) e^{it(\rho^2 - \tau)} b_{\nu, \ell}(\rho) \tau^\alpha \rho^s \rho^{n-1} d\rho dt \right|^2 \frac{d\tau dv}{|z|^{2\beta}} \\ &= \int_X \int_{\mathbb{R}} \left| \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \varphi_{\nu, \ell}(y) \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) \delta(\tau - \rho^2) b_{\nu, \ell}(\rho) \tau^\alpha \rho^s \rho^{n-1} d\rho \right|^2 \frac{d\tau dv}{|z|^{2\beta}}. \end{aligned}$$

Using the delta function definition changing the role of ρ and τ , we are reduced to estimate the integral

$$\begin{aligned} &\int_X \int_0^\infty \left| \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \varphi_{\nu, \ell}(y) (r\sqrt{\rho})^{-\frac{n-2}{2}} J_\nu(r\sqrt{\rho}) b_{\nu, \ell}(\sqrt{\rho}) \rho^{\frac{n-1}{2}} \rho^{\alpha + \frac{s}{2}} \rho^{-\frac{1}{2}} \right|^2 \frac{d\rho dv}{|z|^{2\beta}} \\ &= \int_X \int_0^\infty \left| \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \varphi_{\nu, \ell}(y) (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) b_{\nu, \ell}(\rho) \rho^{2\alpha + s} \rho^{n-2} \right|^2 \frac{\rho d\rho dv}{|z|^{2\beta}} \end{aligned}$$

By the orthogonality, since

$$\int_Y \left| \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \varphi_{\nu, \ell}(y) J_\nu(r\rho) b_{\nu, \ell}(\rho) \right|^2 dy = \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} |J_\nu(r\rho) b_{\nu, \ell}(\rho)|^2$$

we see that the above equals

$$\sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \int_0^\infty \int_0^\infty |(r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) b_{\nu, \ell}(\rho) \rho^{2\alpha + s} \rho^{n-2}|^2 \rho d\rho r^{n-1-2\beta} dr.$$

To estimate it, we make a dyadic decomposition into the integral. Let χ be a smoothing function supported in $[1, 2]$, we see that the above is less than

$$\begin{aligned} (4.3) \quad &\sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \sum_{M \in 2^{\mathbb{Z}}} \int_0^\infty \int_0^\infty \left| (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) b_{\nu, \ell}(\rho) \rho^{2\alpha + s} \rho^{n-2} \chi\left(\frac{\rho}{M}\right) \right|^2 \rho d\rho r^{n-1-2\beta} dr \\ &\lesssim \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \sum_{M \in 2^{\mathbb{Z}}} \sum_{R \in 2^{\mathbb{Z}}} M^{n+4\alpha+2s+2\beta-2} R^{n-1-2\beta} \int_R^{2R} \int_0^\infty \left| (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) b_{\nu, \ell}(M\rho) \chi(\rho) \right|^2 d\rho dr. \end{aligned}$$

Define

$$(4.4) \quad G_{\nu,\ell}(R, M) = \int_R^{2R} \int_0^\infty |(r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) b_{\nu,\ell}(M\rho) \chi(\rho)|^2 d\rho dr.$$

Proposition 4.2. *We have the following inequality*

$$(4.5) \quad G_{\nu,\ell}(R, M) \lesssim \begin{cases} R^{2\nu-n+3} M^{-n} \|b_{\nu,\ell}(\rho) \chi(\frac{\rho}{M}) \rho^{\frac{n-1}{2}}\|_{L^2}^2, & R \lesssim 1; \\ R^{-(n-2)} M^{-n} \|b_{\nu,\ell}(\rho) \chi(\frac{\rho}{M}) \rho^{\frac{n-1}{2}}\|_{L^2}^2, & R \gg 1. \end{cases}$$

Proof. To prove (4.5), we break it into two cases.

• Case 1: $R \lesssim 1$. Since $\rho \sim 1$, thus $r\rho \lesssim 1$. By the property of the Bessel function (2.8), we obtain

$$(4.6) \quad \begin{aligned} G_{\nu,\ell}(R, M) &\lesssim \int_R^{2R} \int_0^\infty \left| \frac{(r\rho)^\nu (r\rho)^{-\frac{n-2}{2}}}{2^\nu \Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} b_{\nu,\ell}(M\rho) \chi(\rho) \right|^2 d\rho dr \\ &\lesssim R^{2\nu-n+3} M^{-n} \|b_{\nu,\ell}(\rho) \chi(\frac{\rho}{M}) \rho^{\frac{n-1}{2}}\|_{L^2}^2. \end{aligned}$$

• Case 2: $R \gg 1$. Since $\rho \sim 1$, thus $r\rho \gg 1$. We estimate by Lemma 2.2 on Bessel function

$$(4.7) \quad \begin{aligned} G_{\nu,\ell}(R, M) &\lesssim R^{-(n-2)} \int_0^\infty |b_{\nu,\ell}(M\rho) \chi(\rho)|^2 \int_R^{2R} |J_\nu(r\rho)|^2 dr d\rho \\ &\lesssim R^{-(n-2)} \int_0^\infty |b_{\nu,\ell}(M\rho) \chi(\rho)|^2 d\rho \lesssim R^{-(n-2)} M^{-n} \|b_{\nu,\ell}(\rho) \chi(\frac{\rho}{M}) \rho^{\frac{n-1}{2}}\|_{L^2}^2. \end{aligned}$$

Thus we prove (4.5). \square

Now we turn to estimate

$$\begin{aligned} &\sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \sum_{M \in 2^\mathbb{Z}} \int_0^\infty \int_0^\infty |(r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) b_{\nu,\ell}(\rho) \rho^{2\alpha+s} \rho^{n-2} \chi(\frac{\rho}{M})|^2 \rho d\rho r^{n-1-2\beta} dr \\ &\lesssim \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \sum_{M \in 2^\mathbb{Z}} \sum_{R \in 2^\mathbb{Z}} M^{n+4\alpha+2s+2\beta-2} R^{n-1-2\beta} G_{\nu,\ell}(R, M) \\ &\lesssim \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \sum_{M \in 2^\mathbb{Z}} \left(\sum_{R \in 2^\mathbb{Z}, R \lesssim 1} M^{n+4\alpha+2s+2\beta-2} R^{n-1-2\beta} R^{2\nu-n+3} M^{-n} \right. \\ &\quad \left. + \sum_{R \in 2^\mathbb{Z}, R \gg 1} M^{n+4\alpha+2s+2\beta-2} R^{n-1-2\beta} R^{-(n-2)} M^{-n} \right) \|b_{\nu,\ell}(\rho) \chi(\frac{\rho}{M}) \rho^{\frac{n-1}{2}}\|_{L^2}^2 \\ &\lesssim \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \sum_{M \in 2^\mathbb{Z}} \left(\sum_{R \in 2^\mathbb{Z}, R \lesssim 1} M^{4\alpha+2s+2\beta-2} R^{2+2\nu-2\beta} \right. \\ &\quad \left. + \sum_{R \in 2^\mathbb{Z}, R \gg 1} M^{4\alpha+2s+2\beta-2} R^{1-2\beta} \right) \|b_{\nu,\ell}(\rho) \chi(\frac{\rho}{M}) \rho^{\frac{n-1}{2}}\|_{L^2}^2. \end{aligned}$$

Note that if $1/2 < \beta < 1 + \nu$ the summations in R converges and further converges to $\|u_0\|_{\dot{H}^{2\alpha+s+\beta-1}(X)}^2$. Hence we prove (4.9). \square

Note $\nu_0 > 0$, we can choose $\alpha = s = 0$ and $\beta = 1$ to obtain

Corollary 4.1. *Let u be the solution of (1.1), then there exists a constant C independent of u_0 such that*

$$(4.8) \quad \| |z|^{-1} u(t, z) \|_{L_t^2(\mathbb{R}; L^2(X))} \leq C \|u_0\|_{L^2(X)}.$$

This corollary is enough for our purpose of this paper, we remark the following additional results due to the independently interesting of local smoothing. We find that the above argument only obtain the result with a small $\epsilon > 0$ loss of regularity

$$\| |z|^{-(1/2+\epsilon)} \mathcal{L}_V^{\frac{1}{4}} u(t, z) \|_{L_t^2(\mathbb{R}; L^2(X))} \leq C \|u_0\|_{\dot{H}^\epsilon(X)}.$$

However if we replace the weight $|z|^{-\beta}$ by $\beta(|z|)$ where β is a smooth function compact supported, we obtain

Corollary 4.2. *Let u be the solution of (1.1) and let $\beta \in C_c^\infty([0, 1])$, then there exists a constant C independent of u_0 such that*

$$(4.9) \quad \|\beta(|z|) \mathcal{L}_V^{\frac{1}{4}} u(t, z) \|_{L_t^2(\mathbb{R}; L^2(X))} \leq C \|u_0\|_{L^2(X)}.$$

Proof. We use the above argument with $\alpha = 0$ and $s = 1/2$ and replace the weight $|z|^{-\beta}$ by $\beta(|z|)$, we will only need to sum in $R \lesssim M$. We modify the argument to obtain

$$\begin{aligned} & \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \sum_{M \in 2^{\mathbb{Z}}} \int_0^\infty \int_0^\infty \left| (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) b_{\nu, \ell}(\rho) \rho^{1/2} \rho^{n-2} \chi\left(\frac{\rho}{M}\right) \right|^2 \rho d\rho r^{n-1} \beta^2(r) dr \\ & \lesssim \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \sum_{M \in 2^{\mathbb{Z}}} \left(\sum_{R \in 2^{\mathbb{Z}}, R \lesssim \min\{1, M\}} M^{n-1} R^{n-1} R^{2\nu-n+3} M^{-n} \right. \\ & \quad \left. + \sum_{R \in 2^{\mathbb{Z}}, 1 \ll R \lesssim M} M^{n-1} R^{n-1} R^{-(n-2)} M^{-n} \right) \|b_{\nu, \ell}(\rho) \chi\left(\frac{\rho}{M}\right) \rho^{\frac{n-1}{2}}\|_{L^2}^2 \\ & \lesssim \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} \sum_{M \in 2^{\mathbb{Z}}} \|b_{\nu, \ell}(\rho) \chi\left(\frac{\rho}{M}\right) \rho^{\frac{n-1}{2}}\|_{L^2}^2 = \|u_0\|_{L^2(X)}^2. \end{aligned}$$

\square

Now we prove Theorem 1.1. By Duhamel formula and Proposition 3.3, we have for $r \geq 2$

$$\begin{aligned} & \|e^{it\mathcal{L}_V} u_0\|_{L_t^q L_z^r} \lesssim \|e^{it\mathcal{L}_V} u_0\|_{L_t^q L_z^{r,2}} \\ & \lesssim \|e^{it\Delta_g} u_0\|_{L_t^q L_z^{r,2}} + \left\| \int_0^t e^{i(t-s)\Delta_g} V(z) e^{is\mathcal{L}_V} u_0 ds \right\|_{L_t^q L_z^{r,2}} \\ (4.10) \quad & \lesssim \|u_0\|_{L_z^2} + \|V(z) e^{is\mathcal{L}_V} u_0\|_{L_t^2 L_z^{\frac{2n}{n+2}, 2}} \\ & \lesssim \|u_0\|_{L_z^2} + \| |z| V(z) \|_{L^{n, \infty}} \| |z|^{-1} e^{is\mathcal{L}_V} u_0 \|_{L_s^2 L_z^2}. \end{aligned}$$

Note that $\|rV(z)\|_{L^{n,\infty}} \leq C\|V_0\|_{L^\infty(Y)}$. By Corollary 4.1, we have the global-in-time local smoothing

$$(4.11) \quad \||z|^{-1}e^{it\mathcal{L}_V}u_0\|_{L_t^2(\mathbb{R};L_z^2(X))} \leq C\|u_0\|_{L^2}.$$

Hence we prove the homogeneous Strichartz estimate (1.3) for all admissible pair (q, r) . By duality, the estimate is equivalent to

$$\left\| \int_{\mathbb{R}} e^{i(t-s)\mathcal{L}_V} F(s) ds \right\|_{L_t^q L_z^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_z^{\tilde{r}'}} ,$$

where both (q, r) and (\tilde{q}, \tilde{r}) satisfy (1.2). By the Christ-Kiselev lemma [7], we obtain for $q > \tilde{q}'$

$$(4.12) \quad \left\| \int_{s < t} e^{i(t-s)\mathcal{L}_V} F(s) ds \right\|_{L_t^q L_z^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_z^{\tilde{r}'}} .$$

Notice that $\tilde{q}' \leq 2 \leq q$, therefore we have proved all inhomogeneous Strichartz estimates except the endpoint $(q, r) = (\tilde{q}, \tilde{r}) = (2, \frac{2n}{n-2})$. On the other hand, we write that

$$u(t, z) = \int_0^t e^{i(t-s)\mathcal{L}_V} F(s) ds = \int_0^t e^{i(t-s)\mathcal{L}_0} (r^{-2}V_0(y)u + F(s)) ds.$$

By the endpoint inhomogeneous Strichartz estimates for Schrödinger equation on Lorentz space in Proposition 3.3, we have

$$\begin{aligned} \|u\|_{L_t^2 L^{\frac{2n}{n-2}}} &\leq \|u\|_{L_t^2 L^{\frac{2n}{n-2}, 2}} \leq C \left(\epsilon \||z|^{-2}u\|_{L^2 L^{\frac{2n}{n+2}, 2}} + \|F\|_{L^2 L^{\frac{2n}{n+2}, 2}} \right) \\ &\leq C \left(\|V_0\|_{L^\infty(Y)} \||z|^{-2}\|_{L^{\frac{n}{2}, \infty}} \|u\|_{L^2 L^{\frac{2n}{n-2}, 2}} + \|F\|_{L^2 L^{\frac{2n}{n+2}, 2}} \right). \end{aligned}$$

If $\|V_0\|_{L^\infty(Y)}$ is small enough such that $C^2\|V_0\|_{L^\infty(Y)} < 1$, then we obtain

$$\|u\|_{L_t^2 L^{\frac{2n}{n-2}, 2}} \leq C\|F\|_{L^2 L^{\frac{2n}{n+2}, 2}}$$

which implies the endpoint inhomogeneous Strichartz estimates when $\|V_0\|_{L^\infty(Y)} \ll 1$. Therefore we prove Theorem 1.1.

5. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2. The key points are the Strichartz estimate in Theorem 1.1 and the Leibniz chain rule in Proposition 2.1.

5.1. Well-posedness theory.

Proposition 5.1 (Local well-posedness theory). *Let $n = 3$. Assume that $u_0 \in H^1(X)$. Then there exists $T = T(\|u_0\|_{H^1}) > 0$ such that the equation (1.5) has a unique solution u with*

$$(5.1) \quad u \in C(I; H^1(X)) \cap L_t^q(I; H_r^1(X)), \quad I = [0, T),$$

where the pair (q, r) is an admissible pair as in (1.2).

Proof. We follow the standard Banach fixed point argument to prove this result. To this end, we consider the map

$$(5.2) \quad \Phi(u(t)) = e^{it\mathcal{L}_V} u_0 - i\gamma \int_0^t e^{i(t-s)\mathcal{L}_V} (|u|^2 u(s)) ds$$

on the complete metric space B_T

$$B_T := \{u \in Y(I) \triangleq C_t(I; H^1) \cap L_t^{q_0}(I; H_{r_0}^1) : \|u\|_{Y(I)} \leq 2CC_1 \|u_0\|_{H^1}\}$$

with the metric $d(u, v) = \|u - v\|_{L_t^{q_0} L_x^{r_0}(I \times X)}$ and (q_0, r_0) satisfies (1.2) and

$$(5.3) \quad (q_0, r_0) = \begin{cases} (5, \frac{30}{11}), & \text{if } \nu_0 > \frac{2}{5}, \\ ((\frac{2}{\nu_0})_+, (\frac{6}{3-2\nu_0})_-) & \text{if } \nu_0 \leq \frac{2}{5}. \end{cases}$$

We need to prove that the operator Φ defined by (5.5) is well-defined on B_T and is a contraction map under the metric d for I .

Let $u \in B_T$. By Sobolev embedding and equivalence of Sobolev spaces,

$$\|u\|_{L_t^{q_0} L_x^6} \leq C \|u\|_{L_t^{q_0} \dot{H}_{r_0}^\theta} \leq C \|u\|_{L_t^{q_0} H_{r_0}^1} \leq \tilde{C} \|u_0\|_{H^1}, \quad \theta = \frac{3}{r_0} - \frac{1}{2}.$$

Then, we have by Strichartz estimate and Proposition 2.1

$$\begin{aligned} \|\Phi(u)\|_{Y(I)} &\leq C \|u_0\|_{H^1} + C \|\langle \mathcal{L}_V^{1/2} \rangle (|u|^2 u)\|_{L_t^2 L_x^{6/5}(I \times X)} \\ &\leq C \|u_0\|_{H^1} + CC_1 |I|^{\frac{1}{2} - \frac{2}{q_0}} \|\langle \mathcal{L}_V^{1/2} \rangle u\|_{L_t^\infty L_x^2} \|u\|_{L_t^{q_0} L_x^6}^2. \end{aligned}$$

Note $\|u\|_{Y(I)} \leq 2CC_1 \|u_0\|_{H^1}$ if $u \in B_T$, we see that for $u \in B_T$,

$$\|\Phi(u)\|_{Y(I)} \leq C \|u_0\|_{H^1} + \tilde{C} |I|^{\frac{1}{2} - \frac{2}{q_0}} (2CC_1 \|u_0\|_{H^1})^3.$$

Taking $|I|$ sufficiently small such that

$$\tilde{C} |I|^{\frac{1}{2} - \frac{2}{q_0}} ((2CC_1 \|u_0\|_{H^1})^3 \leq \frac{1}{2} \|u_0\|_{H^1},$$

we have $\Phi(u) \in B_T$ for $u \in B_T$. On the other hand, by the same argument as before, we have for $u, v \in B_T$,

$$d(\Phi(u), \Phi(v)) \leq C |I|^{\frac{1}{2} - \frac{2}{5}} (\|u\|_{Y(I)}^2 + \|v\|_{Y(I)}^2) d(u, v).$$

Thus we derive by taking $|I|$ small enough

$$d(\Phi(u), \Phi(v)) \leq \frac{1}{2} d(u, v).$$

The standard fixed point argument and applying again the Strichartz estimate gives a unique solution u of (1.5) on $I \times X$ which satisfies the bound (5.1). □

By using Proposition 5.1, mass and energy conservations, we conclude the proof of global well-posed result of Theorem 1.2 in defocusing case $\gamma = 1$.

5.2. Scattering theory. The scattering result of Theorem 1.2 follows from the following Proposition.

Proposition 5.2 (Small data implying scattering). *Let $n = 3$. Assume $\|u_0\|_{H^1(X)} \leq \epsilon$ for a small constant ϵ . Then, there exists a global solution u to (1.5). Moreover, the solution u scatters in sense that there are $u_{\pm} \in H^1(X)$ such that*

$$(5.4) \quad \lim_{t \rightarrow \pm\infty} \|u(t) - e^{it\mathcal{L}_V} u_{\pm}\|_{H^1(X)} = 0.$$

Proof. First, we use the fixed point argument to show the global existence. To do this, we consider the map

$$(5.5) \quad \Phi(u(t)) = e^{it\mathcal{L}_V} u_0 - i\gamma \int_0^t e^{i(t-s)\mathcal{L}_V} (|u|^2 u(s)) ds$$

on the complete metric space B

$$B := \{u \in S^1(\mathbb{R}) : \|u\|_{S^1(\mathbb{R})} \leq 2C\epsilon\}$$

with the metric $d(u, v) = \|u - v\|_{L_t^{q_0} L_x^{r_0}(I \times X)}$, (q_0, r_0) is as in (5.3), and define

$$\|u\|_{S^1(\mathbb{R})} := \sup_{(q,r) \in \Lambda_0; q \geq 2_+} \|u\|_{L_t^q(\mathbb{R}; H_r^1(X))}.$$

Using Strichartz estimate, Proposition 2.1, Hölder's inequality and Sobolev embedding, we get for $u \in B$

$$\begin{aligned} \|u\|_{S^1(\mathbb{R})} &\leq C\|u_0\|_{H^1(X)} + C\||u|^2 u\|_{L_t^2(\mathbb{R}, H_{6/5}^1(X))} \\ &\leq C\|u_0\|_{H^1(X)} + C\|u\|_{L_t^{q_0}(\mathbb{R}, H_{r_0}^1)} \|u\|_{L_t^q(\mathbb{R}, L_x^r)}^2 \\ &\leq C\epsilon + C\|u\|_{L_t^{q_0}(\mathbb{R}, H_{r_0}^1)} \|u\|_{L_t^\infty(\mathbb{R}, L_x^6)} \|u\|_{L_t^{2q}(\mathbb{R}, L_x^{r_1})} \\ &\leq C\epsilon + C\|u\|_{S^1(\mathbb{R})}^3, \end{aligned}$$

where (q_0, r_0) is as in (5.3) and

$$\frac{1}{2} = \frac{1}{q_0} + \frac{2}{q}, \quad \frac{5}{6} = \frac{1}{r_0} + \frac{2}{r}, \quad \frac{2}{r} = \frac{1}{6} + \frac{1}{r_1}.$$

By continuous argument, we obtain

$$\|u\|_{S^1(\mathbb{R})} \leq 2C\epsilon.$$

Hence, we have $\Phi(u) \in B$ for $u \in B$. On the other hand, by the same argument as before, we have for $u, v \in B$,

$$\begin{aligned} d(\Phi(u), \Phi(v)) &\leq C\||u|^2 u - |v|^2 v\|_{L_t^2(\mathbb{R}, L_x^{6/5})} \\ &\leq C\|u - v\|_{L_t^{q_0}(\mathbb{R}, L_x^{r_0})} (\|u\|_{L_t^q(\mathbb{R}, L_x^r)}^2 + \|v\|_{L_t^q(\mathbb{R}, L_x^r)}^2) \\ &\leq C\|(u, v)\|_{S(\mathbb{R})}^2 d(u, v) \\ &\leq C\epsilon^2 \leq \frac{1}{2} d(u, v), \end{aligned}$$

providing that ϵ is sufficient small.

Therefore, the standard fixed point argument gives a unique global solution u of (1.5).

Next, we turn to prove the scattering part. By time reversal symmetry, it suffices to prove this for positive times. For $t > 0$, we will show that $v(t) := e^{-it\mathcal{L}_V}u(t)$ converges in H_x^1 as $t \rightarrow +\infty$, and denote u_+ to be the limit. In fact, we obtain by Duhamel's formula

$$(5.6) \quad v(t) = u_0 - i\gamma \int_0^t e^{-i\tau\mathcal{L}_V}(|u|^2u)(\tau)d\tau.$$

Hence, for $0 < t_1 < t_2$, we have

$$v(t_2) - v(t_1) = -i\gamma \int_{t_1}^{t_2} e^{-i\tau\mathcal{L}_V}(|u|^2u)(\tau)d\tau.$$

Arguing as before, we deduce that

$$\begin{aligned} \|v(t_2) - v(t_1)\|_{H^1(X)} &= \left\| \int_{t_1}^{t_2} e^{-i\tau\mathcal{L}_V}(|u|^{p-1}u)(\tau)d\tau \right\|_{H^1(X)} \\ &\lesssim \| |u|^2u \|_{L_t^2 H_{6/5}^1([t_1, t_2] \times X)} \\ &\lesssim \|u\|_{L_t^{q_0} H_{r_0}^1([t_1, t_2] \times X)} \|u\|_{L_t^q(\mathbb{R}, L_x^r)}^2 \\ &\rightarrow 0 \quad \text{as } t_1, t_2 \rightarrow +\infty. \end{aligned}$$

As t tends to $+\infty$, the limitation of (5.6) is well defined. In particular, we find the asymptotic state

$$u_+ = u_0 - i\gamma \int_0^\infty e^{-i\tau\mathcal{L}_V}(|u|^2u)(\tau)d\tau.$$

Therefore, we conclude the proof of Proposition 5.2. □

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